Problem Set #2, solutions

(1) Consider 1 particle inside the billiard, evolving for a long time τ, during which it makes n collisions with the wall. v = particle speed

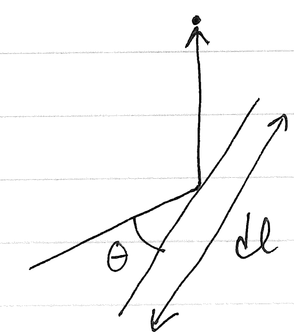
Let r_1 denote the average rate @ which the wall is hit by the particle:

$$r_1 = \frac{n}{\tau} = \frac{v\tau/d}{\tau} = \frac{v}{d}$$

Now consider $N \gg 1$ particles w/ the same speed inside the billiard. The particles don't interact w/ one another. The average collision rate is:

$$r_N = N \cdot r_1 = \frac{Nv}{d} .$$

But we can also compute r_N by considering the average rate of collisions w/ a small segment of the wall:



θ = angle of incidence

rate of collisions w/ this segment

$$= dl \cdot \int_0^{\pi} \left(\rho \frac{d\theta}{2\pi}\right) \cdot v \cdot \sin\theta \qquad = \frac{N v\, dl}{\pi A}$$

particle density ($\rho = N/A$)

flux factor

integrating around the perimeter, we get

$$r_N = \frac{N v l}{\pi A}$$

Setting the two expressions for r_N equal to one another, we get

$$\frac{N v l}{\pi A} = \frac{N v}{d}$$

$$\boxed{d = \frac{\pi A}{l}}$$

Problem 2

(a) $n = \#$ of heads

p_n = probability to observe n heads in N flips

$$= \binom{N}{n} p^n q^{N-n} \approx \frac{N^N}{n^n (N-n)^{N-n}} p^n q^{N-n}$$

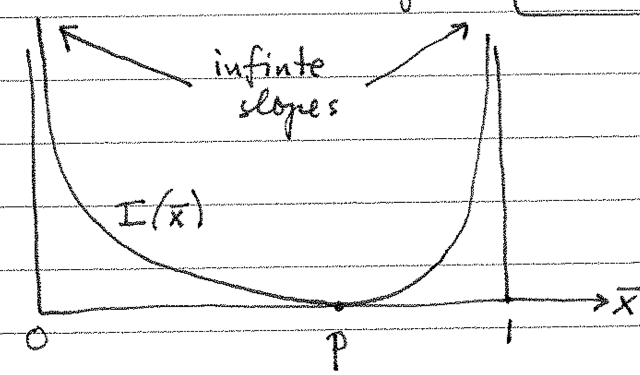
(Stirling formula)

$$= \left[\left(\frac{p}{\bar{x}}\right)^{\bar{x}} \left(\frac{q}{\bar{y}}\right)^{\bar{y}} \right]^N \qquad \bar{x} = n/N = 1 - \bar{y}$$

$$f_N(\bar{x})\, d\bar{x} = p_n\, dn \rightarrow f_N(\bar{x}) = N \cdot p_n$$

$$I(\bar{x}) = \lim_{N\to\infty} -\frac{1}{N} \ln f_N(\bar{x}) = \lim_{N\to\infty} -\frac{1}{N} \ln (N \cdot p_n)$$

$$= \bar{x} \ln\left(\frac{\bar{x}}{p}\right) + \bar{y} \ln\left(\frac{\bar{y}}{q}\right) = \boxed{\bar{x} \ln\left(\frac{\bar{x}}{p}\right) + (1-\bar{x}) \ln \frac{(1-\bar{x})}{(1-p)}}$$



$$I'(\bar{x}) = \ln\left(\frac{\bar{x}}{p}\right) - \ln\left(\frac{1-\bar{x}}{1-p}\right) \quad ; \quad I''(\bar{x}) = \frac{1}{\bar{x}(1-\bar{x})} > 0$$

minimum occurs @ $x = p$: $\begin{cases} I(p) = 0 \\ I'(p) = 0 \\ I''(p) = \dfrac{1}{p(1-p)} \end{cases}$

(b) $\langle e^{\lambda x} \rangle = \int dx\; f(x)\, e^{\lambda x} = q + p e^{\lambda}$

$$g(\lambda) = \ln(q + pe^{\lambda})$$

$$g'(\lambda) = \frac{pe^{\lambda}}{q+pe^{\lambda}} \longrightarrow g'(0) = p$$

$$g''(\lambda) = \frac{qpe^{\lambda}}{(q+pe^{\lambda})^2} \longrightarrow g''(0) = qp$$

$$\mu = \langle x \rangle = \int dx\; f(x)\, x = p = g'(0) \quad \checkmark$$

$$\langle x^2 \rangle = \int dx\; f(x)\, x^2 = p$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = p - p^2 = p(1-p) = pq = g''(0) \quad \checkmark$$

(c) $I(\bar{x}) = \max_{\lambda} \left\{ \lambda \bar{x} - g(\lambda) \right\} \equiv \max_{\lambda} h(\lambda; x)$

$$\frac{\partial h}{\partial \lambda} = \bar{x} - g'(\lambda) = \bar{x} - \frac{pe^{\lambda}}{q+pe^{\lambda}} \overset{\text{set}}{=} 0$$

$$\frac{\partial^2 h}{\partial \lambda} = \frac{-qpe^{\lambda}}{(q+pe^{\lambda})^2} < 0$$

$$\longrightarrow \lambda_{max} = \ln \frac{q\bar{x}}{p(1-\bar{x})} = \ln \frac{q\bar{x}}{p\bar{y}}$$

$$I(\bar{x}) = \lambda_{max}\bar{x} - g(\lambda_{max}) = \bar{x} \ln \frac{q\bar{x}}{p\bar{y}} - \ln \Big(\underbrace{q + p \cdot \frac{q\bar{x}}{p\bar{y}}}_{= q/\bar{y}} \Big)$$

$$= \bar{x} \ln\left(\frac{\bar{x}}{p}\right) - \bar{x} \ln\left(\frac{\bar{y}}{q}\right) - \ln\left(\frac{q}{\bar{y}}\right)$$

$$= \bar{x} \ln\left(\frac{\bar{x}}{p}\right) + \bar{y} \ln\left(\frac{\bar{y}}{q}\right) \quad \checkmark$$

Problem 3

(a) $\Omega(E,V) = \int d\Gamma \;\; \Theta\left[E - \mathcal{H}(\Gamma; V)\right]$

Γ = point in phase space

$= (x_1, y_1, z_1, \theta_1, \varphi_1, \dots, p_{xN}, p_{yN}, p_{zN}, p_{\theta N}, p_{\varphi N})$

Integrating over $dx_1\, dy_1\, dz_1 \dots dx_N\, dy_N\, dz_N$ gives:

$$\Omega = V^N \int_0^{\pi} \sin\theta_1\, d\theta_1 \int_0^{2\pi} d\varphi_1 \cdots \int_0^{\pi} \sin\theta_N\, d\theta_N \int_0^{2\pi} d\varphi_N$$

$$\int dp_{x1} \cdots \int dp_{\varphi N} \;\; \Theta\left[E - K.E.\right]$$

$$= (2\pi V)^N \int_0^{\pi} \sin\theta_1\, d\theta_1 \cdots \int_0^{\pi} \sin\theta_N\, d\theta_N \int d^{5N}p \;\; \Theta\left[E - K.E.\right]$$

integral over all momenta

$$K.E. = \sum_{i=1}^{N}\left(\frac{p_{xi}^2 + p_{yi}^2 + p_{zi}^2}{2m} + \frac{p_{\theta i}^2}{2I} + \frac{p_{\varphi i}^2}{2I \sin^2\theta_i}\right)$$

$$= \sum_{i=1}^{5N}\left(q_1^2 + q_2^2 + \cdots + q_{5N}^2\right) \equiv |\vec{q}\,|^2$$

where $\vec{q} \equiv \left(\frac{p_{x1}}{\sqrt{2m}}, \frac{p_{y1}}{\sqrt{2m}}, \frac{p_{z1}}{\sqrt{2m}}, \frac{p_{\theta 1}}{\sqrt{2I}}, \frac{p_{\varphi 1}}{\sqrt{2I\sin^2\theta_1}}, \dots\right)$

$$\therefore \; d^{5N}p = (2m)^{3N/2}(2I\,\cancel{\sin\theta})^N \, d^{5N}q \cdot \sin\theta_1 \cdots \sin\theta_N$$

So: $\Omega = (2\pi V)^N (2m)^{3N/2} (2I\,\cancel{\sin\theta})^N$

$$\times \underbrace{\left(\int_0^{\pi} \sin^2\theta_1\, d\theta_1\right)}_{\pi/2} \cdots \left(\int_0^{\pi} \sin^2\theta_N\, d\theta_N\right) \int d^{5N}q \;\; \Theta\left(E - |\vec{q}\,|^2\right)$$

$$\Omega = (\pi^2 V)^N (2m)^{3N/2} (2I)^N \underbrace{\int d^{5N}q \;\; \theta\left(E - |\vec{q}\,|^2\right)}_{\substack{\text{volume of } 5N\text{-dimensional} \\ \text{sphere of radius } E^{1/2}}}$$

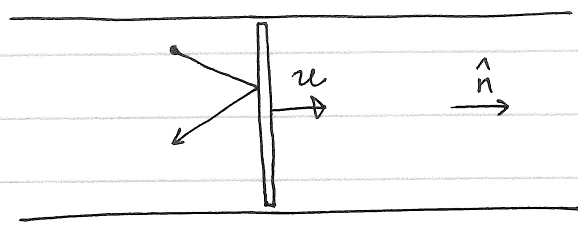
$$= \frac{(E\pi)^{5N/2}}{\frac{5N}{2}\Gamma\left(\frac{5N}{2}\right)} = \frac{(E\pi)^{5N/2}}{\Gamma\left(\frac{5N}{2}+1\right)}$$

$$\Omega = \frac{(\pi^2 V)^N (2m)^{3N/2} (2I)^N (\pi E)^{5N/2}}{\Gamma\left(\frac{5N}{2}+1\right)}$$

$$= f_N \, V^N E^{5N/2}$$

where $f_N = \dfrac{\pi^{9N/2}\, 2^{5N/2}\, m^{3N/2}\, I^N}{\Gamma\left(\frac{5N}{2}+1\right)}$

(4)



velocity of particle $\equiv v\hat{n} + \vec{v}_{\parallel}$

component of velocity parallel to face of piston

during a collision:

$$v\hat{n} + \vec{v}_{\parallel} \rightarrow -(v+2u)\,\hat{n} + \vec{v}_{\parallel} \qquad ; \; v > u$$

$$\Delta E = \frac{m}{2}\left(|\vec{v}_f|^2 - |\vec{v}_i|^2\right) = 2mu\cdot(u-v)$$

σ = area of piston

ρ = # density of particles

$\dot{E}$ = average rate of energy change of gas

$$= \underbrace{\int_u^{\infty} \rho \cdot \underbrace{\sqrt{\frac{\beta m}{2\pi}}\, e^{-\beta m v^2/2}\, dv}_{\text{dist'n of particle speed normal to wall face}} \cdot \underbrace{(v-u)}_{\text{relative particle speed (to wall)}} \cdot \sigma}_{\text{flux of particles w/ normal speed } \in (v,\, v+dv)} \cdot \underbrace{2mu(u-v)}_{\Delta E}$$

$\rightarrow$

$$\dot{E} = -\rho\sigma\sqrt{\frac{\beta m}{2\pi}}\, 2mu \int_u^\infty (v-u)^2 e^{-\beta m v^2/2}\, dv$$

$$= -\rho\sigma\sqrt{\frac{\beta m}{2\pi}}\, 2mu \int_0^\infty (v^2 - 2uv)\, e^{-\beta m v^2/2}\, dv + O(u^3)$$

$$= -2mu\rho\sigma\sqrt{\frac{\beta m}{2\pi}}\left[\frac{1}{4}\sqrt{\pi}\left(\frac{2}{\beta m}\right)^{3/2} - \frac{2u}{\beta m}\right] + O(u^3)$$

$$= -u\rho\sigma\,\beta^{-1} + mu^2\rho\sigma\sqrt{\frac{8}{m\pi\beta}} + O(u^3)$$

$$= -u\rho\sigma k_B T + mu^2\rho\sigma\bar{v} + O(u^3) \quad \checkmark$$

where $\bar{v} = \sqrt{\frac{8}{m\pi\beta}}$ = avg. particle speed (calculated from the Maxwellian velocity distribution)

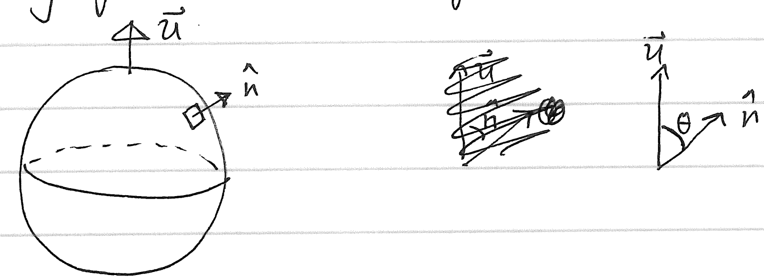
$$\bar{v} = \frac{\int_0^\infty v\, e^{-\beta m v^2/2}\, v^2 dv}{\int_0^\infty e^{-\beta m v^2/2}\, v^2 dv} = \sqrt{\frac{8}{m\pi\beta}}$$

If the piston is moving into the gas, with speed u, then $\Delta E = 2mu(u+v)$ and we get

$$\dot{E} = +u\rho\sigma k_B T + mu^2\rho\sigma\bar{v}$$

If $\hat{n} \not\parallel \vec{u}$, replace u by $\vec{u}\cdot\hat{n}$.

(b) Think of a sphere's surface as consisting of many "pistons" (surface elements)



$$\dot{E} = \underbrace{\oint d\sigma}_{\text{integral over surface of sphere}} \left[\rho\, k_B T\, \hat{n}\cdot\vec{u} + m\bar{v}\rho\,(\hat{n}\cdot\vec{u})^2\right]$$

The first term vanishes by symmetry

$$\left(\oint d\sigma\; \hat{n}\cdot\vec{u} = 0\right) \qquad \hat{n}\cdot\vec{u} = u\cos\theta$$

$$\therefore\ \dot{E} = m\bar{v}\rho\, u^2 \underbrace{\int_0^\infty d\theta \int_0^{2\pi} d\varphi\; R^2 \sin\theta \cdot}_{\text{integration over surface}} \cos^2\theta$$

$$= 2\pi m\bar{v}\rho u^2 R^2 \underbrace{\int_{-1}^{+1} d(\cos\theta)\cdot\cos^2\theta}_{\to\ 2/3}$$

$$= \frac{4\pi}{3} R^2 m\bar{v}\rho u^2 \overset{\text{set}}{=} (\gamma u)\cdot u$$

frictional force

$$\rightarrow\ \boxed{\gamma = \frac{4\pi}{3} R^2 m\bar{v}\rho}$$